Problem 4.39

Verify Equations 4.175 and 4.176 using the Clebsch–Gordan table.

Solution

Equations 4.175 and 4.176 are on page 177.

$$|1\,1\rangle = |\uparrow\uparrow\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|1\;-1\rangle = |\downarrow\downarrow\rangle$$

$$|0\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

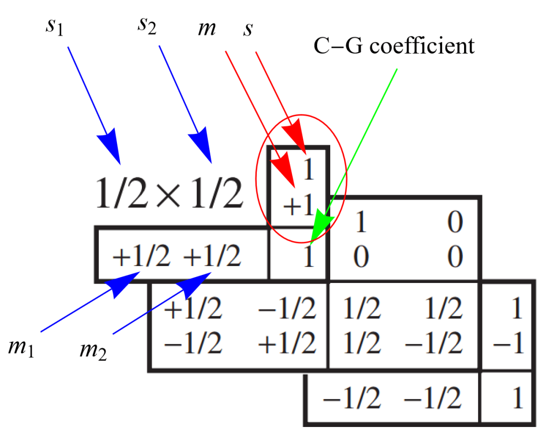
In general, the combined state,

$$|s\,m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1\,s_2\,m_1\,m_2\rangle,$$

with total spin s and z-component m is a linear combination of the composite states $|s_1\,s_2\,m_1\,m_2\rangle$, where $C^{s_1 s_2 s}_{m_1 m_2 m}$ are the Clebsch–Gordon coefficients. Equations 4.175 and 4.176 apply for the case of two spin-1/2 particles, so the relevant table is the one with $1/2 \times 1/2$ at the top left.

1/2 × 1/2	1 +1	1 0	0 0	1 −1
+1/2 +1/2	1			
+1/2 −1/2		1/2	1/2	
−1/2 +1/2		1/2	−1/2	
−1/2 −1/2				1

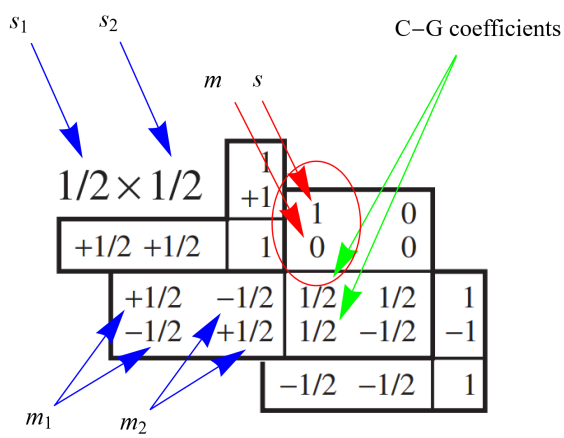
The way to read it for the first combined state, $|1\,1\rangle$, is shown below.



Note that an equivalent way of writing $|s_1\,s_2\,m_1\,m_2\rangle$ is $|s_1\,m_1\rangle|s_2\,m_2\rangle$.

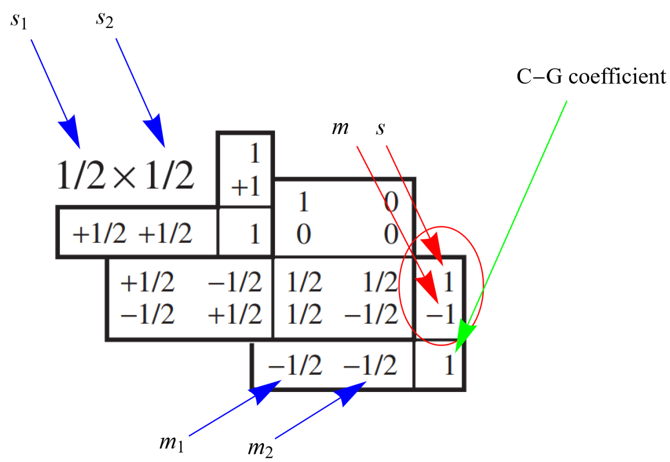
$$\begin{aligned}
|1\,1\rangle &= \sqrt{1}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&= \left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&= \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&= |\uparrow\rangle|\uparrow\rangle \\
&= |\uparrow\uparrow\rangle
\end{aligned}$$

The way to read it for the second combined state, $|1\,0\rangle$, is shown below.



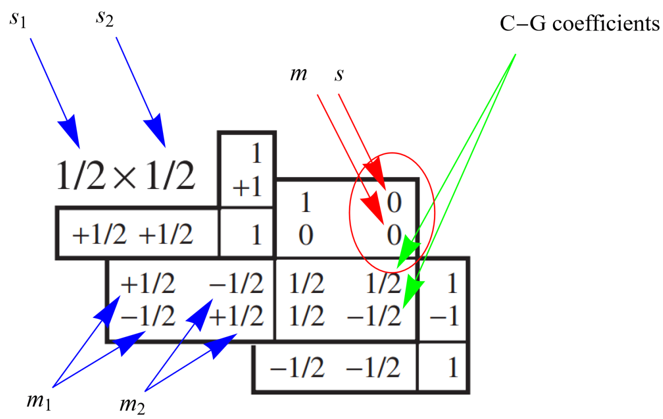
$$\begin{aligned}
|1\,0\rangle &= \sqrt{\frac{1}{2}}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\right\rangle + \sqrt{\frac{1}{2}}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\,\frac{1}{2}\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\right\rangle + \left|\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\,\frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle + \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \\
&= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)
\end{aligned}$$

The way to read it for the third combined state, $|1\ -1\rangle$, is shown below.



$$
\begin{aligned}
|1\ -1\rangle &= \sqrt{1}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\,\frac{-1}{2}\right\rangle \\
&= \left|\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\,\frac{-1}{2}\right\rangle \\
&= \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle \\
&= |\downarrow\rangle|\downarrow\rangle \\
&= |\downarrow\downarrow\rangle
\end{aligned}
$$

The way to read it for the fourth and final combined state, $|0\,0\rangle$, is shown below.



$$
\begin{aligned}
|0\,0\rangle &= \sqrt{\frac{1}{2}}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\right\rangle - \sqrt{\frac{1}{2}}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\,\frac{1}{2}\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\right\rangle - \left|\frac{1}{2}\,\frac{1}{2}\,\frac{-1}{2}\,\frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle - \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)
\end{aligned}
$$