

## Problem 4.39

Verify Equations 4.175 and 4.176 using the Clebsch–Gordan table.

### Solution

Equations 4.175 and 4.176 are on page 177.

$$|1\ 1\rangle = |\uparrow\uparrow\rangle$$

$$|1\ 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|1\ -1\rangle = |\downarrow\downarrow\rangle$$

$$|0\ 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

In general, the combined state,

$$|s\ m\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2 s} |s_1\ s_2\ m_1\ m_2\rangle,$$

with total spin  $s$  and  $z$ -component  $m$  is a linear combination of the composite states  $|s_1\ s_2\ m_1\ m_2\rangle$ , where  $C_{m_1 m_2 m}^{s_1 s_2 s}$  are the Clebsch–Gordan coefficients. Equations 4.175 and 4.176 apply for the case of two spin- $1/2$  particles, so the relevant table is the one with  $1/2 \times 1/2$  at the top left.

				1
			+1	
		1	0	0
	+1/2 +1/2	1	0	0
	+1/2 -1/2	1/2	1/2	1
	-1/2 +1/2	1/2	-1/2	-1
		-1/2 -1/2		1

The way to read it for the first combined state,  $|1\ 1\rangle$ , is shown below.

						1
				+1		
		1	0	0		
	+1/2 +1/2	1	0	0		
	+1/2 -1/2	1/2	1/2	1		
	-1/2 +1/2	1/2	-1/2	-1		
		-1/2 -1/2				1

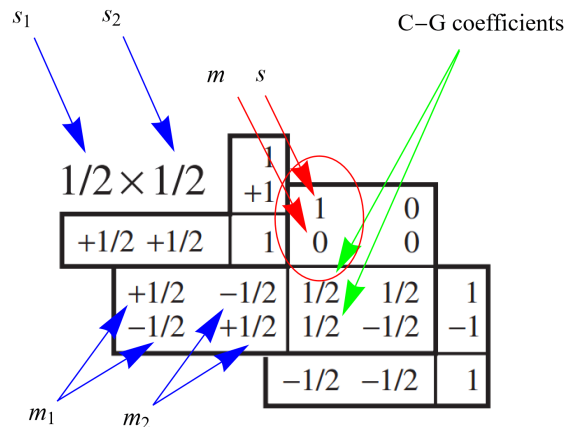
Diagram illustrating the Clebsch–Gordan table for  $1/2 \times 1/2$  particles. The table is annotated with labels and arrows:

- $s_1$  and  $s_2$  (blue arrows) point to the top-left cell  $1/2 \times 1/2$ .
- $m_1$  and  $m_2$  (blue arrows) point to the first two columns of the table.
- $m$  (red arrow) points to the third column.
- $s$  (red arrow) points to the fourth column.
- C-G coefficient (green arrow) points to the value 1 in the top-right cell of the table.

Note that an equivalent way of writing  $|s_1 s_2 m_1 m_2\rangle$  is  $|s_1 m_1\rangle|s_2 m_2\rangle$ .

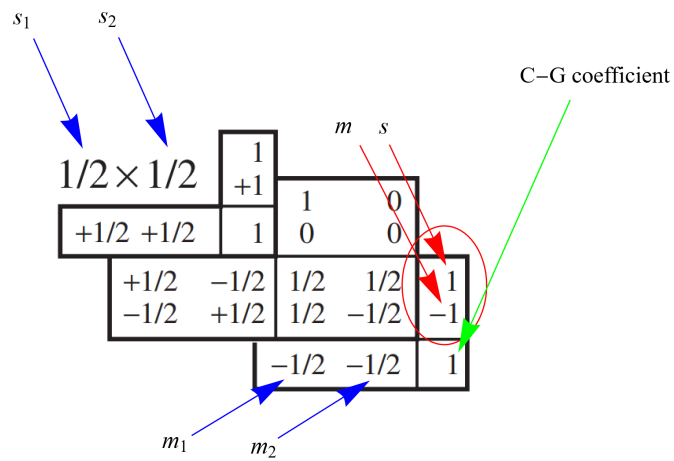
$$\begin{aligned}
 |11\rangle &= \sqrt{1} \left| \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right\rangle \\
 &= \left| \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right\rangle \\
 &= \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \\
 &= |\uparrow\rangle|\uparrow\rangle \\
 &= |\uparrow\uparrow\rangle
 \end{aligned}$$

The way to read it for the second combined state,  $|10\rangle$ , is shown below.



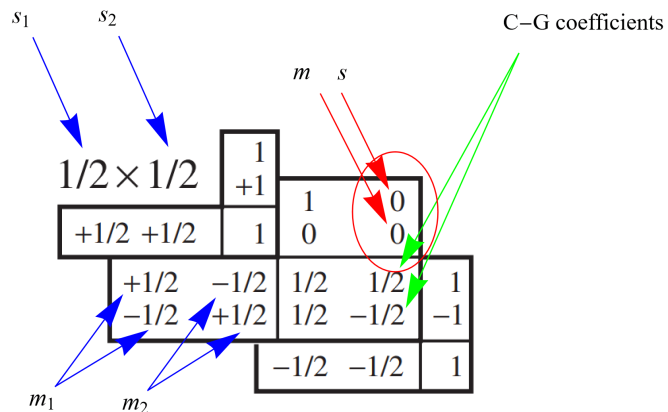
$$\begin{aligned}
 |10\rangle &= \sqrt{\frac{1}{2}} \left| \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{-1}{2} \right\rangle + \sqrt{\frac{1}{2}} \left| \frac{1}{2} \frac{1}{2} \frac{-1}{2} \frac{1}{2} \right\rangle \\
 &= \frac{1}{\sqrt{2}} \left( \left| \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{-1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} \frac{-1}{2} \frac{1}{2} \right\rangle \right) \\
 &= \frac{1}{\sqrt{2}} \left( \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{-1}{2} \right\rangle + \left| \frac{1}{2} \frac{-1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right) \\
 &= \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \\
 &= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)
 \end{aligned}$$

The way to read it for the third combined state,  $|1 -1\rangle$ , is shown below.



$$\begin{aligned}
 |1 -1\rangle &= \sqrt{1} \left| \frac{1}{2} \frac{1}{2} \frac{-1}{2} \frac{-1}{2} \right\rangle \\
 &= \left| \frac{1}{2} \frac{1}{2} \frac{-1}{2} \frac{-1}{2} \right\rangle \\
 &= \left| \frac{1}{2} \frac{-1}{2} \right\rangle \left| \frac{1}{2} \frac{-1}{2} \right\rangle \\
 &= |\downarrow\rangle |\downarrow\rangle \\
 &= |\downarrow\downarrow\rangle
 \end{aligned}$$

The way to read it for the fourth and final combined state,  $|00\rangle$ , is shown below.



$$\begin{aligned}
 |00\rangle &= \frac{1}{\sqrt{2}} \left| \begin{matrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 2 & 2 \end{matrix} \right\rangle - \frac{1}{\sqrt{2}} \left| \begin{matrix} 1 & 1 & -1 & 1 \\ 2 & 2 & 2 & 2 \end{matrix} \right\rangle \\
 &= \frac{1}{\sqrt{2}} \left( \left| \begin{matrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 2 & 2 \end{matrix} \right\rangle - \left| \begin{matrix} 1 & 1 & -1 & 1 \\ 2 & 2 & 2 & 2 \end{matrix} \right\rangle \right) \\
 &= \frac{1}{\sqrt{2}} \left( \left| \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \right\rangle \left| \begin{matrix} 1 & -1 \\ 2 & 2 \end{matrix} \right\rangle - \left| \begin{matrix} 1 & -1 \\ 2 & 2 \end{matrix} \right\rangle \left| \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \right\rangle \right) \\
 &= \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \\
 &= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
 \end{aligned}$$